Wiener-Hopf operators induced by multipliers

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1. Introduction. Let R denote the real line $(-\infty, \infty)$ and let $M(R)$ denote the commutative Banach algebra of complex valued Borel measures on R equipped with total variation norm and with multiplication defined by convolution of measures. Let $R^+=[0, \infty)$ and $R^-=(-\infty, 0]$. For each $1 \leqq p \leqq \infty$, let $L^p(R)$ (resp. $L^p(R^+)$, $L^p(R^-)$) denote the usual Lebesgue space of complex valued Borel measurable functions on R (resp. R^+, R^-). *To avoid unnecessary repetition, it will be assumed henceforth that the index p of any L^p-space under consideration satisfies the constraint* $1 \leqq p \leqq \infty$. The subspace of $M(R)$ consisting of those measures whose support is contained in R^+ (resp. R^-) will be denoted by $M(R^+)$ (resp. $M(R^-)$). We shall frequently identify $L^p(R^+)$ and $L^p(R^-)$ as subspaces of $L^p(R)$. We write I (resp. I_+, I_-) for the identity operator on $L^p(R)$ (resp. $L^p(R^+)$, $L^p(R^-)$) and P (resp. Q) for the natural projection of $L^p(R)$ onto $L^p(R^+)$ (resp. $L^p(R^-)$). If $1 \leqq p < \infty$ (resp. $p=\infty$), we write $B(L^p(R))$ for the space of continuous (resp. weak*-continuous) linear operators on $L^p(R)$ equipped with the usual operator norm.

If $\mu \in M(R)$ and $f \in L^p(R)$, then the convolution

$$[\mu * f](x) = \int f(x-t)\, d\mu(t)$$

defines a.e. an element $\mu * f \in L^p(R)$ with $\|\mu * f\|_p \leqq \|\mu\| \, \|f\|_p$. For each $\mu \in M(R)$ the operator $S(\mu, p) \in B(L^p(R))$ is defined by

$$S(\mu, p)f = \mu * f, \quad f \in L^p(R),$$

and $\|S(\mu, p)\| \leqq \|\mu\|$. We say that $S(\mu, p)$ is the *convolution operator* on $L^p(R)$ induced by μ.

If T is any operator on $L^p(R)$, the operator $\mathrm{pr}\,(T)$ on $L^p(R^+)$ is defined by

$$\mathrm{pr}\,(T)f = PTf, \quad f \in L^p(R^+).$$

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If $T\in B(L^p(R))$, then $\mathrm{pr}\,(T)\in B(L^p(R^+))$ and $\|\mathrm{pr}\,(T)\|\leqq\|T\|$. For each $\mu\in M(R)$, the *Wiener—Hopf operator* $W(\mu, p)$ on $L^p(R^+)$ induced by μ is defined by $W(\mu, p)= =\mathrm{pr}\,(S(\mu, p))$.

Several authors have considered the Wiener—Hopf operators induced by various classes of measures (cf. [8], [12]), particularly with regard to their inversion. R. G. Douglas and J. L. Taylor [4] have recently provided inversion criteria of great generality. We summarize their results in the following

Theorem 1.1. *If $\mu\in M(R)$, $W(\mu, 1)$ is invertible if and only if $\mu\in\exp(M(R))$. If μ is invertible in $M(R)$, $W(\mu, p)$ is invertible if and only if $\mu\in\exp(M(R))$.*

An important consequence of this result is that the invertibility of $W(\mu, p)$ is independent of the index p provided that μ is invertible in $M(R)$. Douglas and Taylor also show that this need not be the case if μ is not invertible. Specifically, they exhibit a noninvertible measure $\nu\in M(R^+)$ for which $W(\nu, 2)$ is invertible. This example motivates our consideration of the more general class of Wiener—Hopf operators induced by multipliers. As we shall subsequently show, the invertibility of $W(\nu, 2)$ implies the invertibility of $S(\nu, 2)$. Since ν is not invertible in $M(R)$, $S(\nu, 2)^{-1}$ is a multiplier (cf. Section 2) which is not a convolution operator and, moreover, $W(\nu, 2)^{-1}$ is $\mathrm{pr}\, S(\nu, 2)^{-1}$.

In the next section we provide a summary of some important facts concerning Fourier transforms, multipliers and pseudomeasures which we will need later. In § 3 we define the class of Wiener—Hopf operators induced by multipliers and prove theorems analogous to results such as those of Hartman—Winter, Coburn and Brown—Halmos in the theory of Toeplitz operators. In § 4 we examine the Wiener—Hopf factorization technique for the inversion of a Wiener—Hopf operator. In contrast to the results of previous authors, we give an example of an invertible Wiener—Hopf operator on $L^2(R^+)$ whose inverse cannot be expressed in the form W_+W_- where W_+ and W_- are analytic and coanalytic (cf. Section 2) Wiener—Hopf operators respectively.

In § 5 we conclude by considering the problem of interpolating the inverse of a Wiener—Hopf operator suggested by the example of Douglas and Taylor. The Wiener—Pitt measure is used to provide an example of a Wiener—Hopf operator $W(\omega, p)$ which is invertible for $1<p<\infty$ yet not invertible for $p=1, \infty$. Although we are unable to show that interpolation of the inverse occurs in the general case, we show that interpolation does occur when the Wiener—Hopf operator is either analytic or coanalytic.

2. Fourier transforms, multipliers and pseudomeasures. If $f\in L^1(R)$, we define the Fourier transform of f by

$$\hat{f}(x)=\int e^{ixt}f(t)\,dt, \quad x\in R.$$

If p' denotes the conjugate exponent of p and $1 \leqq p \leqq 2$, then the Fourier transform defines a bounded linear mapping of $L^1(R) \cap L^p(R)$ (equipped with the norm from $L^p(R)$) into $L^{p'}(R)$. The unique continuous extension of this mapping from $L^p(R)$ into $L^{p'}(R)$ will also be called the Fourier transform and the result of applying this mapping to an element $f \in L^p(R)$ will be denoted by $\hat{f}$. In the case $p=2$ the Fourier transform is an invertible operator on $L^2(R)$ and we shall denote by U the *Fourier—Plancheral transform* on $L^2(R)$ defined by $Uf=(2\pi)^{-1/2}\hat{f}$. The operator U is a unitary operator on $L^2(R)$.

We shall eventually have need of relations between the Fourier transform and the Hardy spaces $H^p(R)$. The definitions and basic facts concerning these spaces may be found in [6] and [10]. The following result does not seem to be explicitly stated in the standard references. Since we will make crucial use of it, we sketch the proof.

Theorem 2.1. *If* $1 \leqq p \leqq 2$ *and* $f \in L^p(R^+)$, *then* $\hat{f} \in H^{p'}(R)$. *Moreover,* $U(L^2(R^+)) = H^2(R)$.

Proof. Let π^+ be the upper half-plane $\{z \mid \operatorname{Im} z > 0\}$ and define the Laplace transform of f by

$$Lf(z) = \int e^{izt} f(t)\,dt, \quad z \in \pi^+.$$

The function Lf is analytic in π^+. Consider the family of functions f_y, $y>0$, defined by $f_y(t) = e^{-yt} f(t)$ and note that $\sup_{y>0} \|f_y\|_p = \|f\|_p$ and that $\lim_{y\to 0} \|f_y - f\|_p = 0$. Since $Lf(x+iy) = \hat{f}_y(x)$, it follows that $Lf \in H^p(\pi^+)$ and that $\hat{f}$ is the boundary function for Lf. This proves the first assertion. The second assertion is then just a form of the Paley—Wiener theorem [15, Theorem 19.2, p. 368]. Q.E.D.

The space $M(R)$ and the spaces $L^p(R)$ admit natural involutions defined by setting $\mu^*(E) = \overline{\mu(-E)}$ for $\mu \in M(R)$ and E a Borel set in R and by setting $f^*(x) = \overline{f(-x)}$ a.e. for $f \in L^p(R)$. If the Fourier—Stieltjes transform of $\mu \in M(R)$ is defined by $\hat{\mu}(x) = \int e^{ixt}\,d\mu(t)$, then these involutions have the following properties:

$$\hat{\mu}^* = \bar{\hat{\mu}}, \quad \mu \in M(R),$$

and

$$\hat{f}^* = \bar{\hat{f}}, \quad f \in L^p(R),\ 1 \leqq p \leqq 2.$$

For each $a \in R$, let δ_a denote the measure with positive unit mass at the point a. Just as the ambiguity of space for the projections P and Q causes no problems, so we shall frequently write S_a in place of $S(\delta_a, p)$ and W_a in place of $W(\delta_a, p)$. We define a *multiplier* on $L^p(R)$ to be an operator $S \in B(L^p(R))$ such that S commutes with S_a for each $a \in R$. The set of multipliers on $L^p(R)$ will be denoted by $\mathcal{M}^p$.

It is clear that $\mathcal{M}^p$ is an inverse-closed algebra of operators on $L^p(R)$ and that $\mathcal{M}^p$ contains the convolution operators on $L^p(R)$. If $p=1$ or $p=\infty$, then $\mathcal{M}^p$ is precisely the set of convolution operators on $L^p(R)$ [13, Theorem 3.1.1 and Theorem 3.4.1]. If the adjoint operator on $L^{p'}(R)$ of an operator on $L^p(R)$, $1 \leqq p < \infty$, is defined by means of the pairing

$$(f, g) = \int f\bar{g}\,dx,\ f \in L^p(R),\ g \in L^{p'}(R),$$

then a simple application of Fubini's theorem shows that $S(\mu, p)^* = S(\mu^*, p')$ for $\mu \in M(R)$ and $1 \leqq p < \infty$. Setting $\mu = \delta_a$ we see that $S_a^* = S_{-a}$ and that $\mathcal{M}^{p^*} = \mathcal{M}^{p'}$ if $1 \leqq p < \infty$.

Let $A(R) = \{\hat{f} \mid f \in L^1(R)\}$ and let $A(R)$ be given the induced norm from $L^1(R)$. The algebra $A(R)$ is then a Banach algebra and is isometrically isomorphic to $L^1(R)$. The Banach space dual of $A(R)$ will be denoted by $P(R)$ and the elements of this dual space will be called *pseudomeasures.* The natural isomorphism mapping $\sigma \to \hat{\sigma}$ of $P(R)$ onto $L^\infty(R)$ will be called the Fourier transform on $P(R)$. If $\sigma \in P(R)$, the element $\hat{\sigma} \in L^\infty(R)$ is uniquely determined by the relation

$$\sigma(\hat{f}) = \int \hat{\sigma} f\,dx, \quad f \in L^1(R).$$

The space $P(R)$ is a commutative C^*-algebra via the induced operations from $L^\infty(R)$. If $M(R)$ is identified as a subalgebra of $P(R)$ by means of the relation

$$\mu(\hat{f}) = \int \hat{f}\,d\mu, \quad \mu \in M(R), f \in L^1(R),$$

then the multiplication, involution and Fourier transform defined on $P(R)$ are consistent with those previously defined on $M(R)$. In particular, we may denote by $*$ the multiplication on $P(R)$.

Theorem 2.2. [13, Theorem 4.3.1] *The relation*

$$(Sf)\hat{} = \hat{\sigma}\hat{f}, \quad f \in L^2(R),$$

between elements $S \in \mathcal{M}^2$ and $\sigma \in P(R)$ determines an isometric algebraic isomorphism between $\mathcal{M}^2$ and $P(R)$.

For $1 \leqq p \leqq \infty$, let $\lambda(p) = |(p-2)/p|$. The value of $\lambda(p)$ may be regarded as a measure of the distance of p from 2 and the function $\lambda(\cdot)$ is symmetric with respect to conjugate indices. The next result is essentially contained in [13, pp. 95—97].

Theorem 2.3. *If $S \in \mathcal{M}^p$ and $\lambda(r) \leqq \lambda(p)$, then S maps $L^1(R) \cap L^\infty(R)$ into $L^p(R) \cap L^{p'}(R)$ and hence into $L^r(R)$. Moreover, the restriction of S to $L^1(R) \cap L^\infty(R)$ has a unique extension to an element of $\mathcal{M}^r$. The resulting mapping of $\mathcal{M}^p$ into $\mathcal{M}^r$ is an injective norm-decreasing algebra homomorphism and is an isometric isomorphism if $\lambda(r) = \lambda(p)$.*

Combining theorems 2.2 and 2.3 we see that $\mathcal{M}^p$ may be identified with a subalgebra of $\mathcal{M}^2$ and hence with a subalgebra of $P(R)$ (containing $M(R)$). In particular, to each multiplier on $L^p(R)$ there is associated a unique pseudomeasure. The notation $S(\sigma, p)$ will be used to denote the multiplier on $L^p(R)$, *if it exists*, having σ as its associated pseudomeasure. With this notation, the natural mapping of $\mathcal{M}^p$ into $\mathcal{M}^r$ for $\lambda(r) \leqq \lambda(p)$ is given by $S(\sigma, p) \to S(\sigma, r)$.

As an easy consequence of theorems 2.2 and 2.3 we have

Theorem 2.4. *If* $1 \leqq p \leqq 2$ *and* $S = S(\sigma, p)$, *then*

$$(Sf)\hat{} = \hat{\sigma}\hat{f}, \quad f \in L^p(R).$$

If $\sigma \in P(R)$, we say that σ is *analytic* if $\hat{\sigma} \in H^\infty(R)$ and that σ is *coanalytic* if $\hat{\sigma} \in \overline{H^\infty(R)}$. If the support of a pseudomeasure is defined as in [7], then a pseudomeasure is analytic (resp. coanalytic) if and only if its support is contained in R^+ (resp. R^-). If $S \in \mathcal{M}^p$, we say that S is *analytic* (resp. *coanalytic*) if S leaves $L^p(R^+)$ (resp. $L^p(R^-)$) invariant. Theorems 2.1, 2.2 and 2.3 imply the following

Theorem 2.5. *If* $S = S(\sigma, p)$, *then* S *is analytic (resp. coanalytic) if and only if* σ *is analytic (resp. coanalytic).*

3. The class $\mathcal{W}^p$. If $W \in B(L^p(R^+))$, we say that W is a *Wiener—Hopf operator* if $W = \mathrm{pr}\,(S)$ for some $S \in \mathcal{M}^p$. The class of Wiener—Hopf operators on $L^p(R^+)$ will be denoted by $\mathcal{W}^p$. If $S = S(\sigma, p)$, the Wiener—Hopf operator $\mathrm{pr}\,(S)$ induced by S may be denoted by $W(\sigma, p)$. Since the mapping $T \to \mathrm{pr}\,(T)$ of $B(L^p(R))$ into $B(L^p(R^+))$ is linear, $\mathrm{pr}\,(I) = I_+$ and $\mathrm{pr}\,(T)^* = \mathrm{pr}\,(T^*)$ if $1 \leqq p < \infty$, it follows that $\mathcal{W}^p$ is a linear subspace of $B(L^p(R^+))$, $I_+ \in \mathcal{W}^p$ and $\mathcal{W}^{p*} = \mathcal{W}^{p'}$ if $1 \leqq p < \infty$.

In the case $p = 1$ we know that $\mathcal{M}^1$ consists precisely of the convolution operators $S(\mu, 1)$ for measures $\mu \in M(R)$. Thus $\mathcal{W}^1$ consists precisely of the Wiener—Hopf operators $W(\mu, 1)$ induced by measures $\mu \in M(R)$. By duality, a similar statement holds for $\mathcal{W}^\infty$. If, for the moment, we assume that each $W \in \mathcal{W}^p$ is induced by a unique $S \in \mathcal{M}^p$ (a fact that will be established later), then it follows from Theorem 2.3 that we may identify $\mathcal{W}^p$ as a subspace of $\mathcal{W}^r$ whenever $\lambda(r) \leqq \lambda(p)$. In particular, we may think of $\mathcal{W}^p$ as being contained in $\mathcal{W}^2$, depending symmetrically on the index p and growing larger as p approaches 2.

If $\varphi \in L^\infty(R)$, define the operator $M_\varphi \in B(L^2(R))$ by setting $M_\varphi f = \varphi f$ for each $f \in L^2(R)$. Let P_+ be the orthogonal projection of $L^2(R)$ onto $H^2(R)$. The Toeplitz operator $T_\varphi \in B(H^2(R))$ induced by $\varphi \in L^\infty(R)$ is defined by setting $T_\varphi f = P_+ M_\varphi f$ for each $f \in H^2(R)$. We now assert that *the class* $\mathcal{W}^2$ *is unitarily equivalent to the class of Toeplitz operators on* $H^2(R)$. For suppose that $\sigma \in P(R)$ and $f \in L^2(R^+)$ and let U_0 be the restriction of the Fourier—Plancherel transform to $L^2(R^+)$. Applying Theorems 2.1 and 2.2 we have

$$U_0 W(\sigma, 2) f = UPS(\sigma, 2) f = P_+ US(\sigma, 2) f = P_+ M_{\hat{\sigma}} U_0 f = T_{\hat{\sigma}} U_0 f.$$

Thus $U_0 W(\sigma, 2) U_0^{-1} = T_{\hat{\sigma}}$ and the assertion follows by observing that $\hat{\sigma}$ ranges over $L^\infty(R)$ as σ ranges over $P(R)$.

The following result gives a simple characterization of the Wiener—Hopf operators on $L^p(R^+)$ analogous to a well known characterization of Toeplitz operators [1, Theorem 6].

Theorem 3.1. *If $W\in B(L^p(R^+))$, then a necessary and sufficient condition in order that $W\in\mathcal{W}^p$ is that $W_{-a}WW_a=W$ for $a\geqq 0$.*

Proof. For each $a\in R$, let P_a denote the natural projection of $L^p(R)$ onto $L^p([a,\infty))$. If $W\in\mathcal{W}^p$, so that $W=\mathrm{pr}\,(S)$ for some $S\in\mathcal{M}^p$, then for each $a\geqq 0$ and $f\in L^p(R^+)$ we have

$$W_{-a}WW_af = P(S_{-a}P)S(PS_a)f = P(P_{-a}S_{-a})S(S_aP_{-a})f = PSf = Wf.$$

This establishes the necessity of the condition.

Since the assertion in the case $p=\infty$ follows by duality from the case $p=1$, it follows that we need only prove the sufficiency of the condition in the case $1\leqq p<\infty$. So suppose that $W\in B(L^p(R^+))$ where $1\leqq p<\infty$ and that $W_{-a}WW_a=W$ for each $a\geqq 0$. Regard R as a directed set with its natural order and consider the net $\{S_{-a}WPS_a\}_{a\in R}$ in $B(L^p(R))$. If $a\leqq b$ and $f\in L^p([-a,\infty))$, then

$$P_{-a}S_{-b}WPS_bf = P_{-a}S_{-a}S_{(a-b)}WPS_{(b-a)}S_af =$$
$$= S_{-a}PS_{(a-b)}WPS_{(b-a)}S_af = S_{-a}W_{(a-b)}WW_{(b-a)}S_af = S_{-a}WPS_af.$$

Since the net $\{S_{-a}WPS_a\}_{a\in R}$ is bounded in norm and $1\leqq p<\infty$, it follows that this net is strongly convergent on the set $\bigcup_{a\in R} L^p(-a,\infty))$. Since the latter set is dense in $L^p(R)$, it follows that the net is strongly convergent on $L^p(R)$ to some $S\in B(L^p(R))$. For each $b\in R$,

$$S_bS = s-\lim_a (S_bS_{-a}WPS_a) = s-\lim_a (S_{(b-a)}WPS_{(a-b)}S_b) = SS_b.$$

Thus $S\in\mathcal{M}^p$. If $f\in L^p(R^+)$, then

$$PSf = \lim_a (PS_{-a}WPS_af) = \lim_a (W_{-a}WW_af) = Wf.$$

Thus $W=\mathrm{pr}\,(S)$ and $W\in\mathcal{W}^p$.

Theorem 3.2. *If $1\leqq p<\infty$, $S\in\mathcal{M}^p$ and $W=\mathrm{pr}\,(S)$, then $S=s-\lim_a S_{-a}WPS_a$.*

Proof. Since $1\leqq p<\infty$, $s-\lim_a P_{-a}=I$. The desired conclusion follows from the fact that, for each $a\in R$,

$$S_{-a}WPS_a = S_{-a}PSPS_a = P_{-a}S_{-a}SS_aP_{-a} = P_{-a}SP_{-a}. \qquad \text{Q.E.D.}$$

An important consequence of this result is that each Wiener—Hopf operator on $L^p(R^+)$, $1\leqq p<\infty$, is induced by a unique multiplier on $L^p(R)$ and that pr is isometric on $\mathcal{M}^p$. By duality, the same is true in the case $p=\infty$.

Corollary 3.3. *For each* $W\in\mathcal{W}^p$ *there is a unique* $S\in\mathcal{M}^p$ *such that* $W=\operatorname{pr}(S)$ *and, moreover,* $\|W\|=\|S\|$.

Corollary 3.4. *If* $1\leqq p<\infty$, $S\in\mathcal{M}^p$ *and* $W=\operatorname{pr}(S)$, *then*

$$\inf_{\substack{f\in L^p(R^+)\\ \|f\|_p=1}}\|Wf\|_p\leqq\inf_{\substack{g\in L^p(R)\\ \|g\|_p=1}}\|Sg\|_p.$$

Proof. Let $g\in L^p(R)$ and $\|g\|_p=1$. Since $1\leqq p<\infty$, $\lim_a\|PS_ag\|_p=1$ so that, by Theorem 3.2,

$$\|Sg\|_p=\lim_a\|S_{-a}WPS_ag\|_p=\lim_a\|WPS_ag\|_p\geqq\inf_{\substack{f\in L^p(R^+)\\ \|f\|_p=1}}\|Wf\|_p.$$

The assertion now follows immediately. Q.E.D.

The next result is an analogue for Wiener—Hopf operators of the spectral inclusion theorem of HARTMAN and WINTNER [9] for Toeplitz operators.

Theorem 3.5. *If* $S\in\mathcal{M}^p$, $S=S(\sigma,p)$ *and* $W=\operatorname{pr}(S)$, *then*

$$\text{ess range }\hat{\sigma}\subseteqq\operatorname{sp}(S)\subseteqq\operatorname{sp}(W).$$

Proof. By Theorem 2.3, $\operatorname{sp}(S(\sigma,2))\subseteqq\operatorname{sp}(S(\sigma,p))$ and, by Theorem 2.2, $S(\sigma,2)=U^{-1}M_{\hat{\sigma}}U$ so that $\operatorname{sp}(S(\sigma,2))=$ess range $\hat{\sigma}$. Thus ess range $\hat{\sigma}\subseteqq\operatorname{sp}(S)$.

To prove the second inclusion, it will suffice to show that S is invertible whenever W is invertible. So suppose that W is invertible. If $p=1$, then $\sigma\in M(R)$ and, by Theorem 1.1, $\sigma\in\exp(M(R))$. This implies that σ is invertible in $M(R)$ and that S is invertible. If $p=\infty$, the same conclusion follows by duality. Finally, if $1<p<\infty$, then the invertibility of S follows from Corollary 3.4 and the fact [5, Lemma 3 on p. 488] that an operator on a Banach space is invertible if and only if both the operator and its adjoint are bounded from below. Q.E.D.

If $W\in\mathcal{W}^p$, we know from Corollary 3.3 that W is induced by a unique multiplier $S\in\mathcal{M}^p$. We may therefore make the following definition: if $W\in\mathcal{W}^p$, we say that W is *analytic* (resp. *coanalytic*) if its inducing multiplier is analytic (resp. coanalytic). If $W=W(\sigma,p)$, it follows from Theorem 2.5 that W is analytic or coanalytic according as σ is analytic or coanalytic. Theorem 2.5 also implies that a multiplier or Wiener—Hopf operator which is both analytic and coanalytic is a scalar multiple of I or I_+ respectively.

We now turn to a consideration of the multiplicative properties of the class $\mathcal{W}^p$. If W_1, $W_2\in\mathcal{W}^p$ and either W_2 is analytic or W_1 is coanalytic, then $W_1W_2\in\mathcal{W}^p$. For if S_i is the multiplier inducing W_i, $i=1,2$, it is easily seen that

$$\operatorname{pr}(S_1)\operatorname{pr}(S_2)=\operatorname{pr}(S_1S_2),$$

so that under these conditions W_1W_2 is the Wiener—Hopf operator induced by S_1S_2. Conversely, the stated conditions are necessary in order that the product W_1W_2 be a Wiener—Hopf operator.

Theorem 3.6. *Let* $S_i \in \mathcal{M}^p$ *and* $W_i = \operatorname{pr}(S_i)$, $i=1, 2, 3$. *In order that* $W_1 W_2 = W_3$ *it is necessary and sufficient that* $S_1 S_2 = S_3$ *and that either* W_2 *be analytic or* W_1 *be coanalytic.*

This result is the analogue for Wiener—Hopf operators of a well known theorem of Brown and Halmos [1, Theorem 8] for Toeplitz operators. Although the proof of Theorem 3.6 can be achieved by first reducing to the class $\mathcal{W}^2$ and then applying the theorem of Brown and Halmos to the corresponding Toeplitz operators, we prefer to give an independent proof. (The proof in [1] uses the existence of an orthonormal basis in Hilbert space to reduce to a matrix computation.) We require the following

Lemma 3.7. *Let* $1 \leqq p < \infty$, K_0 *be a right translation invariant subspace of* $L^p(R^-)$ *and let* K *be the smallest closed left translation invariant subspace of* $L^p(R^-)$ *containing* K_0. *Then either* $K_0 = \{0\}$ *or* $K = L^p(R^-)$.

Proof. The hypothesis that K_0 is right translation invariant means that K_0 is invariant under the operators QS_a for $a \geqq 0$. It is clear, then, that $\chi_{[-a,0]} f \in K$ whenever $f \in K_0$ and $a > 0$. Also, if $f \in K_0$, then almost every $x \in R^-$ is a p-th order Lebesgue point for f so that the condition

$$\lim_{h \to 0} \left(\frac{1}{h} \int_0^h |f(x-t) - f(x)|^p \, dt \right)^{1/p} = 0 \tag{3.1}$$

holds for almost every $x \in R^-$.

Suppose, now, that $K_0 \neq \{0\}$. From the hypothesis on K_0 it follows that we can choose some $f \in K_0$ such that $f(0) = 1$ and (3.1) holds for $x = 0$. In order to show that $K = L^p(R^-)$ it suffices to show that K contains the function $\chi_{[-a,0]}$ for each $a > 0$. So let $a > 0$ and consider the sequence of functions $\{g_n\}_{n=1}^{\infty}$ in $L^p(R^-)$, where

$$g_n = \sum_{k=0}^{n-1} S_{-ka/n}(\chi_{[-a/n,0]} f).$$

It is easy to see that the sequence $\{g_n\}_{n=1}^{\infty}$ is in K and that $\lim_{n\to\infty} \|g_n - \chi_{[-a,0]}\|_p = 0$. Since K is closed, it follows that $\chi_{[-a,0]} \in K$. Q.E.D.

Proof of Theorem 3.6. The remarks preceding the statement of the theorem show the sufficiency of the conditions and it remains to show that they are necessary. Moreover, the necessity of the conditions in the case $p = \infty$ follows from their necessity when $p = 1$ so that we need only consider the case $1 \leqq p < \infty$.

Suppose that $1 \leqq p < \infty$ and that $W_1 W_2 = W_3$. Since

$$(S_{-a} W_1 P S_a)(S_{-a} W_2 P S_a) = S_{-a} W_1 W_2 P S_a, \quad a \in R,$$

it follows from Theorem 3.2 that $S_1 S_2 = S_3$. Let K_0 be the range of QS_2P and $K = \{f \in L^p(R^-) \mid PS_1 f = 0\}$. It is clear that K_0 is a right translation invariant subspace of $L^p(R^-)$ and that K is a closed left translation invariant subspace of $L^p(R^-)$. Since $S_1 S_2 = S_3$, the hypothesis that $W_1 W_2 = W_3$ implies that $PS_1 PS_2 P = PS_1 S_2 P$. It follows that $PS_1 QS_2 P = 0$ so that K contains K_0. By Lemma 3.7, either $K_0 = \{0\}$ or $K = L^p(R^-)$. Thus, either S_2 is analytic or S_1 is coanalytic. Q.E.D.

Corollary 3.8. *A necessary and sufficient condition in order that an operator $W \in B(L^p(R^+))$ be an analytic (resp. coanalytic) Wiener—Hopf operator is that W commute with W_a (resp. W_{-a}) for $a \geqq 0$.*

Proof. By symmetry it is enough to prove the assertion in the analytic case. The necessity of the condition is immediate. So suppose, conversely, that $W \in B(L^p(R^+))$ and that $W_a W = W W_a$ for each $a \geqq 0$. Then $W = W_{-a} W_a W = W_{-a} W W_a$ for $a \geqq 0$ so that, by Theorem 3.1, $W \in \mathscr{W}^p$. Since W_a is analytic but not coanalytic if $a > 0$, Theorem 3.6 and the equality $W_a W = W W_a$ imply that W is analytic. Q.E.D.

Corollary 3.9. *Let $S \in \mathscr{M}^p$, $W = \mathrm{pr}\,(S)$ and S be analytic (resp. coanalytic). Then a necessary and sufficient condition in order that W be invertible is that S have an analytic (resp. coanalytic) inverse. If the condition is satisfied, $W^{-1} = \mathrm{pr}\,(S^{-1})$.*

Proof. It is enough to consider the analytic case. Moreover, the sufficiency of the condition and the last assertion follow immediately from Theorem 3.6. So suppose, conversely, that S is analytic and that W is invertible. By Theorem 3.5, S is invertible. Since $S(L^p(R^+)) = W(L^p(R^+)) = L^p(R^+)$, it follows that S^{-1} is analytic. Q.E.D.

If T_φ is a non-zero Toeplitz operator on $H^2(R)$, Coburn [2] has shown that T_φ either has trivial kernel or dense range. The existence of an analogous result for Wiener—Hopf operators (induced by measures) was conjectured by Douglas and Taylor [4]. The following theorem establishes such an analogue for Wiener—Hopf operators.

Theorem 3.10. *If $W \in \mathscr{W}^p$ and $W \neq 0$, then W either has trivial kernel or dense range (w^*-dense range if $p = \infty$).*

Proof. By duality, it is enough to prove the assertion in the case $1 \leqq p \leqq 2$. So suppose that $1 \leqq p \leqq 2$, $W \in \mathscr{W}^p$ and that W has non-trivial kernel and non-dense range. We will show that $W = 0$.

Let $W = \mathrm{pr}\,(S)$ where $S = S(\sigma, p)$. Since W has non-trivial kernel, we may choose $f \in L^p(R^+)$ such that $f \neq 0$ and $Sf \in L^p(R^-)$. Since W has non-dense range, $W^* = \mathrm{pr}\,(S^*)$ has non-trivial kernel. Thus we may choose $g \in L^{p'}(R^+)$ such that $g \neq 0$ and $S^* g \in L^{p'}(R^-)$. Since Sf, $f^* \in L^p(R)$ and $S^* g$, $g \in L^{p'}(R)$, the convolutions $Sf * g^*$ and $f^* * S^* g$ are well defined continuous functions on R. Moreover, a straightforward computation shows that $Sf * g^* = (f^* * S^* g)^*$. Since both $Sf * g^*$ and $f^* * S^* g$ vanish on R^+, it follows that $Sf * g^* = 0$. Since $g^* \neq 0$ in $L^{p'}(R^-)$, we may apply Titchmarsh's convolution theorem [16, Theorem 153] to conclude that $Sf = 0$

in $L^p(R)$. (We are grateful to Professor Donald Sarason for suggesting this use of the Titchmarsh convolution theorem to us.) By Theorem 2.4, $\hat{\sigma}\hat{f}=0$ in $L^{p'}(R)$ and, by Theorem 2.1, $\hat{f}\in H^{p'}(R)$. Since $f\neq 0$ in $L^p(R^+)$, $\hat{f}\neq 0$ in $H^{p'}(R)$ [11, p.142] and therefore $\hat{f}$ is non-zero a.e. [10, p. 133]. It follows that $\hat{\sigma}=0$ in $L^\infty(R)$. Thus $\sigma=0$ in $P(R)$ and $W=0$. Q.E.D.

4. The failure of factorization. The problem of finding conditions under which a Wiener—Hopf operator W will be invertible and, when the inverse exists, of providing an analytical representation for W^{-1}, has been of central importance in the theory of Wiener—Hopf operators. The principal tool for inverting a Wiener—Hopf operator has been the so-called *Wiener—Hopf technique*, first developed by N. Wiener and E. Hopf [17] in a somewhat different setting and applied by several subsequent authors to the Wiener—Hopf operators induced by various classes of measures [4], [8], [12]. In this section we will first describe the Wiener—Hopf technique in its general form and then provide an example showing its inadequacy — at least in the case $p=2$.

Let $S\in\mathcal{M}^p$ and suppose that S can be factored in the form $S=S_-S_+$ where $S_+, S_-\in\mathcal{M}^p$, S_+ is analytic and S_- is coanalytic. Then by Theorem 3.6, $\mathrm{pr}\,(S)=\mathrm{pr}\,(S_-)\,\mathrm{pr}\,(S_+)$. If, moreover, S_+ has an analytic inverse and S_- has a coanalitic inverse, then Corollary 3.9 implies that $\mathrm{pr}\,(S_+)$ and $\mathrm{pr}\,(S_-)$ are invertible and that

$$\mathrm{pr}\,(S)^{-1} = \mathrm{pr}\,(S_+^{-1})\,\mathrm{pr}\,(S_-^{-1}). \tag{4.1}$$

Formula (4.1) is called the *Wiener—Hopf formula* and suggests the following definition: if $W\in\mathcal{W}^p$ and W is invertible, we say that W^{-1} is *factorable* if there exist $W_+, W_-\in\mathcal{W}^p$ with W_+ analytic and W_- coanalytic such that $W^{-1}=W_+W_-$.

The case of Wiener—Hopf operators induced by measures is of particular interest. If $\mu\in\exp\big(M(R)\big)$, then $\mu=\exp(\nu)$ for some $\nu\in M(R)$ and we can write $\nu=\nu_-+\nu_+$ where $\nu_\pm\in M(R^\pm)$. (This decomposition need not be unique.) Thus $\mu=\exp(\nu_-)*\exp(\nu_+)$, $\exp(\nu_\pm)\in M(R^\pm)$ and $\exp(\nu_\pm)^{-1}=\exp(-\nu_\pm)\in M(R^\pm)$. We therefore have

$$W(\mu, p)^{-1} = W\big(\exp(-\nu_+), p\big)W\big(\exp(-\nu_-), p\big).$$

Thus an exponential measure induces an invertible Wiener—Hopf operator whose inverse is factorable. The following result is of interest in relation to Theorem 1.1.

Theorem 4.1. *If $\mu\in M(R)$ and $W(\mu, p)$ is invertible, then a necessary and sufficient condition in order that $W(\mu, p)^{-1}$ be factorable with factors induced by measures is that $\mu\in\exp\big(M(R)\big)$.*

Proof. The sufficiency of the condition has already been shown (see also [4]). Conversely, suppose that $W(\mu, p)^{-1}=W(\nu_+, p)\,W(\nu_-, p)$ for some $\nu_\pm\in M(R^\pm)$. Then

$$W(\mu, p)W(\nu_+, p)W(\nu_-, p) = I_+ = W(\delta_0, p).$$

By Theorem 3.6, we conclude that

$$W(\mu * \nu_+, p) W(\nu_-, p) = W(\delta_0, p).$$

Applying Theorem 3.6 yet again, we conclude that $\mu * \nu_+ \in M(R^-)$, $\mu * \nu_+ * \nu_- = \delta_0$ and that μ is invertible in $M(R)$. Since μ is invertible in $M(R)$, Theorem 1.1 implies that $\mu \in \exp(M(R))$. Q.E.D.

In view of the above, it is somewhat surprising that factorization should fail in the case $p=2$. For if $S \in \mathcal{M}^2$ and $W = \mathrm{pr}\,(S)$ is invertible, then we know that S must be invertible. Since $\mathcal{M}^2$ is isometrically and algebraically isomorphic to $P(R)$ and hence to $L^\infty(R)$, it follows that *the invertible elements in $\mathcal{M}^2$ are the same as the exponentials in $\mathcal{M}^2$*. Thus S has a logarithm $S(\sigma, 2)$ in $\mathcal{M}^2$. What goes wrong is that, in contrast to the case of measures, it may not be possible to write σ as the sum of an analytic and a coanalytic pseudomeasure [10, p. 151]. We shall not, however, base our example upon this fact, since logarithms in $\mathcal{M}^2$ are highly non-unique.

One further remark is in order. Factorization can be restored in the case $p=2$ provided that we do not require that the factors be induced by multipliers. If $S \in \mathcal{M}^2$ and $W = \mathrm{pr}\,(S)$ is invertible, then S is invertible and a theorem of Devinatz and Shinbrot [3, Theorem 5] implies that there exist invertible *operators* A_+ and A_- on $L^2(R)$ such that $S = A_- A_+$, $A_+(L^2(R^+)) = L^2(R^+)$, $A_-(L^2(R^-)) = L^2(R^-)$ and $W^{-1} = \mathrm{pr}\,(A_+^{-1})\,\mathrm{pr}\,(A_-^{-1})$.

If $f \in L^p(R)$, then in order that f have the same modulus as some nonzero element of $H^p(R)$ it is necessary and sufficient [10, p. 133] that

$$\int \frac{\log |f|}{1+t^2}\, dt > -\infty.$$

In constructing our example showing the failure of factorization in the case $p=2$ we shall make use of the fact that the argument of a nonzero element of $H^p(R)$ is, likewise, not arbitrary. To simplify matters, we introduce an auxiliary mapping. Let $Z(R)$ denote the multiplicative group of measurable functions on R which are nonzero a.e. and define the mapping u of $Z(R)$ into itself by setting $u(f) = f/|f|$ for each $f \in Z(R)$. For each, $f, g \in Z(R)$ we have

$$\text{(i)} \qquad u(fg) = u(f)u(g),$$

$$\text{(ii)} \qquad u(f^{-1}) = u(f)^{-1} = u(\bar{f}) = \overline{u(f)}.$$

$$\text{(iii)} \qquad u(u(f)) = u(f).$$

Note that $Z(R)$ contains $H^p(R) - \{0\}$.

Lemma 4.2. *If $f \in H^\infty(R) - \{0\}$, then $u(f^{-1}) = u(g)$ for some $g \in H^p(R) - \{0\}$ if and only if $f^{-1} \in H^p(R)$.*

Proof. Suppose that $f\in H^\infty(R)-\{0\}$ and that $u(f^{-1})=u(g)$ for some $g\in H^p(R)-\{0\}$. Then $u(fg)=u(f)u(g)=u(f)u(f)^{-1}=1$. Since $fg\in H^p(R)$, it follows that $fg=c$ for some positive constant c. Thus $f^{-1}=c^{-1}g\in H^p(R)$. Q.E.D.

Our example may now be constructed as follows: Let f be the continuous branch of $(x+i)^{1/3}$ on R with $0<\arg(f)<\pi/3$. Then $f\notin H^\infty(R)$ and $f^{-1}\in H^\infty(R)$ so that, by Lemma 4.2,

$$u(f)\neq u(g) \quad \text{for each} \quad g\in H^\infty(R)-\{0\}. \tag{4.2}$$

Since $0<\arg u(f)<\pi/3$, the closed convex hull of the essential range of $u(f)$ does not contain 0. From this it follows [1, p. 99] that the Toeplitz operator $T_{u(f)}$ is invertible on $H^2(R)$. Let σ be the pseudomeasure with $\hat{\sigma}=u(f)$ and let $W=W(\sigma, 2)$. Since W is unitarily equivalent to $T_{u(f)}$, W is invertible on $L^2(R^+)$.

Suppose, now, that W^{-1} is factorable. Then $W^{-1}=W(\sigma_+, 2)\,W(\sigma_-, 2)$ for some $\sigma_+, \sigma_-\in P(R)$ with σ_+ analytic and σ_- coanalytic. The equations

$$WW(\sigma_+, 2)W(\sigma_-, 2) = I_+ = W(\delta_0, 2), \quad W(\sigma_+, 2)W(\sigma, 2)W = I_+ = W(\delta_0, 2)$$

imply, as in the proof of Theorem 4.1, that $\sigma*\sigma_+*\sigma_-=\delta_0$, that $\sigma_-*\sigma$ is an analytic inverse for σ_+ and that $\sigma*\sigma_+$ is a coanalytic inverse for σ_-. From this it follows that $u(f)=f_1\bar{f}_2$ for some $f_1, f_2\in H^\infty(R)$ with $f_1^{-1}, f_2^{-1}\in H^\infty(R)$. Then $u(f)=$ $=u(u(f))=u(f_1)u(\bar{f}_2)=u(f_1)u(f_2^{-1})=u(f_1f_2^{-1})$. Since $f_1f_2^{-1}\in H^\infty(R)-\{0\}$, this contradicts (4.2) and we conclude that W^{-1} is not factorable.

Before turning to the next section we comment on the example of Douglas and Taylor [4] mentioned in § 1. In this example, a noninvertible measure ν is constructed with the property that $\hat{\nu}$ and $\hat{\nu}^{-1}$ are in $H^\infty(R)$. It follows that ν has an analytic inverse $\sigma\in P(R)$ and that $S(\nu, 2)^{-1}=S(\sigma, 2)$. Thus, by Corollary 3.9, $W(\nu, 2)^{-1}=$ $=W(\sigma, 2)$ so that $W(\nu, 2)^{-1}$ is (trivially) factorable.

5. Interpolation of the inverse. We begin this section by exhibiting a measure ω such that $W(\omega, p)$ is invertible for $1<p<\infty$ yet not invertible for $p=1, \infty$. It is a consequence of the work of Krein [12] and of Gohberg and Feldman [8] that such a measure must necessarily have a nonzero singular continuous part. Our example is based on the fact [14, p. 107] that there exists a continuous positive measure $\nu\in M(R)$ such that $\|\nu\|=1$, $\nu^*=\nu$ and $\pm i\in\operatorname{sp}(\nu)$. Let ν be such a measure and let $\omega=\delta_0+\nu^2$. The measure ω has the remarkable property that $\hat{\omega}=1+|\hat{\nu}|^2\geqq 1$ yet ω is not invertible in $M(R)$. Wiener and Pitt [18, Theorem 3] were the first to show the existence of measures exhibiting such spectral misbehavior and we shall therefore call ω the *Wiener—Pitt measure.*

Since ω is not invertible in $M(R)$, Theorem 1.1 implies that $W(\omega, 1)$ is not invertible. Since $\omega^*=\omega$, it follows that $W(\omega, \infty)$ is not invertible.

To show that $W(\omega, p)$ is invertible for $1<p<\infty$, it suffices to show that $W(\omega, p)$ is invertible for $1<p\leqq 2$. So let $1<p\leqq 2$ and set $\lambda=\lambda(p)$ so that $\frac{1}{p}=\lambda\frac{1}{1}+(1-\lambda)\frac{1}{2}$. Since $\hat{\omega}\geqq 1$ we may choose $0<\varepsilon<1$ so that $\|1-\varepsilon\hat{\omega}\|_\infty<1$. By the Riesz interpolation theorem [5, p. 525] and the Hölder inequality we have

$$\|I_+-\varepsilon W(\omega, p)\| = \|W(\delta_0-\varepsilon\omega, p)\| \leqq \|W(\delta_0-\varepsilon\omega, 1\|^{\lambda}\|W(\delta_0-\varepsilon\omega, 2)\|^{1-\lambda} \leqq$$
$$\leqq \lambda\|W(\delta_0-\varepsilon\omega, 1)\|+(1-\lambda)\|W(\delta_0-\varepsilon\omega, 2)\|.$$

By Corollary 3.3 and [13, Cor. 0.1.1] we have

$$\|W(\delta_0-\varepsilon\omega, 1)\| = \|S(\delta_0-\varepsilon\omega, 1)\| = \|\delta_0-\varepsilon\omega\|$$

and, by Corollary 3.3 and Theorem 2.2, we have

$$\|W(\delta_0-\varepsilon\omega, 2)\| = \|S(\delta_0-\varepsilon\omega, 2)\| = \|1-\varepsilon\hat{\omega}\|_\infty.$$

It follows that

$$\|I_+-\varepsilon W(\omega, p)\| < \lambda\|\delta_0-\varepsilon\omega\|+(1-\lambda) = \lambda\|(1-\varepsilon)\delta_0-\varepsilon v^2\|+(1-\lambda) \leqq$$
$$\leqq \lambda\big((1-\varepsilon)+\varepsilon\big)+(1-\lambda) = 1,$$

so that $W(\omega, p)$ is invertible.

By a considerable refinement of the argument just given we can prove the following.

Theorem 5.1. *If $\mu\in M(R)$ and K is the closed convex hull of the range of $\hat{\mu}$, then*

$$\operatorname{sp}\big(W(\mu, p)\big) \subseteqq \{z\,|\operatorname{dist}(z, K) \leqq \lambda(p)\|\mu'\|\},$$

where μ' denotes the measure $\mu-\mu(\{0\})\delta_0$.

Proof. Since $(\mu-z\delta_0)\hat{} =\hat{\mu}-z$ and $(\mu-z\delta_0)'=\mu'$, it is enough to prove that $W(\mu, p)$ is invertible for those p such that $\lambda(p)\|\mu'\|<\operatorname{dist}(0, K)$. If $0\in K$ there is nothing to prove, so suppose that $\operatorname{dist}(0, K)>0$ and that $\lambda(p)\|\mu'\|<\operatorname{dist}(0, K)$. By Parseval's formula we see that

$$\mu(\{0\}) = \lim_{a\to\infty}\int\frac{\operatorname{Sin}(ax)}{ax}\,d\mu(x) = \lim_{a\to\infty}\frac{1}{2a}\int_{-a}^{a}\hat{\mu}(x)\,dx,$$

so that $\mu(\{0\})\in K$ and hence $\mu(\{0\})\neq 0$. Without loss of generality we may assume that $\mu(\{0\})=1\in K$. Since K is a compact convex set, it follows that (for a suitable branch of $\arg(z)$) $\arg(K)=[\theta_1, \theta_2]$ where $\theta_1\leqq 0\leqq\theta_2$ and $\theta_2-\theta_1<\pi$ and that there exists a complex number z_0 with $|z_0|=1$ such that

$$\inf_{k\in K}\operatorname{Re}(\bar{z}_0 k) = \operatorname{dist}(0, K).$$

Since $\arg(K)=[\theta_1, \theta_2]$, it is evident that $z_0=\exp(i\theta_0)$ for some $\theta_0\in[\theta_1, \theta_2]$ satisfying

$$-\pi/2 < \theta_1-\theta_0 \leqq \theta_2-\theta_0 < \pi/2.$$

Since $\mu(\{0\})=1$, $\mu=\delta_0+\mu'$ and for each $\varepsilon>0$ we have

$$\|z_0\delta_0-\varepsilon\mu\| = \|(z_0-\varepsilon)\delta_0-\varepsilon\mu'\| = |z_0-\varepsilon|+\varepsilon\|\mu'\|.$$

It follows from elementary calculus that

$$\|z_0\delta_0-\varepsilon\mu\| = 1+\varepsilon\|\mu'\|-\varepsilon\operatorname{Cos}\theta_0+o(\varepsilon). \tag{5.1}$$

Now let $r_1=\inf_{k\in K}\operatorname{Re}(\bar z_0 k)$ and $r_2=\sup_{k\in K}\operatorname{Re}(\bar z_0 k)$. Since $\arg(K)=[\theta_1,\theta_2]$, it follows from the usual equation for a line in polar coordinates that

$$\sup_{k\in K}|z_0-\varepsilon k| = \sup_{k\in K}|1-\varepsilon\bar z_0 k|$$

$$\leqq \sup_{r_1\leqq r\leqq r_2}\ \sup_{\theta_1\leqq\theta\leqq\theta_2}|1-\varepsilon e^{-i\theta_0}r\operatorname{Sec}(\theta-\theta_0)e^{i\theta}| \leqq \sup_{r_1\leqq r\leqq r_2}|1-\varepsilon r+i\varepsilon rm|,$$

where $m=\max\{\operatorname{Tan}(\theta_0-\theta_1),\operatorname{Tan}(\theta_2-\theta_0)\}$. By elementary calculus it follows that for $\varepsilon>0$ sufficiently small

$$\sup_{k\in K}|z_0-\varepsilon k| \leqq |1-\varepsilon r_1+i\varepsilon r_1 m|,$$

from which it follows that

$$\|z_0-\varepsilon\hat\mu\|_\infty \leqq 1-\varepsilon r_1+o(\varepsilon). \tag{5.2}$$

By the same reasoning as in our example using the Wiener—Pitt measure we conclude from (5.1) and (5.2) that

$$\|z_0 I_+-\varepsilon W(\mu,p)\| \leqq \lambda(p)\{1+\varepsilon\|\mu'\|-\varepsilon\operatorname{Cos}\theta_0\}+\big(1-\lambda(p)\big)\{1-\varepsilon r_1\}+o(\varepsilon).$$

Since $1\in K$ it follows that $r_1\leqq\operatorname{Cos}\theta_0$ and that

$$\|z_0 I_+-\varepsilon W(\mu,p)\| \leqq 1+\varepsilon\big(\lambda(p)\,\|\mu'\|-r_1\big)+o(\varepsilon).$$

Since $\lambda(p)\|\mu'\|<\operatorname{dist}(0,K)=r_1$, it follows that $\|z_0 I_+-\varepsilon W(\mu,p)\|<1$ for some $\varepsilon>0$ and hence that $W(\mu,p)$ is invertible. Q.E.D.

In view of Theorem 2.3, it seems natural to conjecture that if a Wiener—Hopf operator $W(\sigma,p)$ is invertible on $L^p(R^+)$, then the Wiener—Hopf operators $W(\sigma,r)$ for $\lambda(r)\leqq\lambda(p)$ are also invertible. If $\sigma\in M(R)$ and σ has no singular continuous part then the conjecture is true even without the restriction on r (cf. [8]). If $p=1$ or $p=\infty$, then the conjecture is also true and is a consequence of Theorem 1.1. Our example employing the Wiener—Pitt measure gives further support for this conjecture. We have been unable to prove this conjecture — even in the case of measures. However, it is quite easy to prove in the case of an analytic or coanalytic Wiener—Hopf operator. In a sense, therefore, the behavior exhibited in the example of Douglas and Taylor is typical of at least these two classes of Wiener—Hopf operators.

Theorem 5.2. *If $W(\sigma,p)$ is an analytic or coanalytic Wiener—Hopf operator and $W(\sigma,p)$ is invertible, then $W(\sigma,r)$ is invertible for $\lambda(r)\leqq\lambda(p)$.*

Proof. It suffices to give the proof in the analytic case. If $W(\sigma, p)$ is an invertible analytic Wiener—Hopf operator, then Corollary 3.9 implies that $S(\sigma, p)$ has an analytic inverse and we see that $S(\sigma, p)^{-1} = S(\sigma^{-1}, p)$ where σ^{-1} is the analytic inverse of σ in $P(R)$. If $\lambda(r) \leqq \lambda(p)$, then, by Theorem 2.3, $S(\sigma, r)$ is invertible and $S(\sigma, r)^{-1} = S(\sigma^{-1}, r)$. Since $S(\sigma^{-1}, r)$ is analytic, Corollary 3.9 implies that $W(\sigma, r)$ is invertible. Q.E.D.

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